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Removable Singularities of Solutions of Systems of Linear Partial Differential Equations with Real Analytic Coefficients

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0. Introduction.

On the mental basis of all the studies on continuation of solutions there lies Hartogs' continuation theorem asserting that a compact (a fortiori isolated) singularity of a holomorphic function of several variables is always removable. As is well known this result has been interpreted as the theorem on continuation of solutions of overdetermined systems with constant coefficients by S. Bochner, L. Ehrenpreis, B. Malgrange, V. P. Palamodov, H. Komatsu etc. (see Palamodov[1] and Ehrenpreis[1] for more detailed references). However, these results on overdetermined systems concern continuation of "general" solutions: If we limit ourselves to the consideration of sufficiently regular solutions from the beginning, we obtain much wider classes of systems of equations which admit such extension of solutions. For example, an isolated weak singularity of a C^∞ solution is always removable if and only if the system contains no hypoelliptic component (Palamodov[1]), and a compact singularity of a real analyitc solution is always removable if and only if the system contains no elliptic component (Kaneko[1]). Recently Abramczuk[1] has shown that the last condition by Kaneko is also necessary and sufficient for a compact singularity of a Denjoy Carleman (quasi-analytic) class solution to be always removable (On the contrary the exact condition for the removability of general isolated singularities of C^∞ solutions is not yet known even for a single equation with constant coefficients).

The present article aims to extend these studies to the case of systems of equations with real analytic coefficients. A result corresponding to the overdetermined systems is already given by Kawai[1]. In the case of variable coefficients, however, even a single (hence far from overdetermined) equation may be exempt from solutions with isolated singularities as is shown there. As for the continuation of regular solutions of single equations with variable coefficients we have already given some results in Kaneko [2], [3], [5] based on the estimation of the singular spectrum of boundary values. Extending these results we will give here some classes of systems of equations which are free from regular solutions with "small" singularities. Note that Kawai[1], [2] also give some results on the extension of real analytic solutions of a sytem with variable coefficients based on the consideration of the generalized Levi form.

To familiarize the reader with the notion of systems we have given a rather detailed preparation of necessary terms.

1. Study of system of equations with real analytic coefficients.

First we review about necessary knowledge on a system $\mathcal{M}$ of differential equations with real analytic coefficients. As general references we cite Björk[1], Kashiwara[1] (Unfortunately there is yet no complete translation of this fundamental work of Kashiwara[1]. We can, however, refer to good references on a system of micro-differential equations written in English, such as Björk[1], Kashiwara [2], S-K-K[1], and from these we can well imagine the case of systems of differential equations.).

System of differential equations. A system $\mathcal{M}$ is by definition a right $\mathfrak{D}$-module with finite representation:

$$O \leftarrow \mathcal{M} \leftarrow \mathfrak{D}^{r_1} \xleftarrow{{}^tP(x,D)} \mathfrak{D}^{r_2}, \tag{1.1}$$

where $\mathfrak{D}$ is the sheaf of rings of germs of differential operators with real analytic coefficients on $\boldsymbol{R}^n$, $P(x,D)$ is an (r_2, r_1)-matrix of differential operators which we assume is globally defined on a neighborhood of a fixed closed ball $B_\delta = \{|x| \leq \delta\} \subset \boldsymbol{R}^n$ and ${}^tP$ is its dual operator. In the sequel we also let $\mathcal{M}$ or $\mathfrak{D}$ denote their stalks at the origin or the corresponding modules of their global sections on the ball B_δ. Since the sheaf of rings $\mathfrak{D}$ is coherent and its section module on the ball is Noetherian (Kashiwara[1], Proposition 1.1.5 and Proposition 1.1.3) this abuse does not cause problem.

Since the above therminology is not necessarily familiar to the specialists of partial differential equations we here give its intuitive explanation. What we usually call a system of equations is the matrix $P(x,D)$. The quotient module $\mathcal{M}$ serves as an intrinsic object corresponding to all the systems equivalent to one particular representation $P(x,D)$. In fact, two systems $P(x,D)$ and $Q(x,D)$ satisfying $\mathcal{M}=\mathfrak{D}^{r_1}/{}^tP(x,D)\mathfrak{D}^{r_2}\cong\mathcal{N}=\mathfrak{D}^{s_1}/{}^tQ(x,D)\mathfrak{D}^{s_2}$ have the same solutions in the sense that choosing a commutative diagram

$$\begin{array}{ccccccc} O\leftarrow & \mathcal{M} & \leftarrow & \mathfrak{D}^{r_1} & \xleftarrow{\;{}^tP(x,D)\;} & \mathfrak{D}^{r_2} \\ & \wr\downarrow & & \downarrow R & & \downarrow S \\ O\leftarrow & \mathcal{N} & \leftarrow & \mathfrak{D}^{s_1} & \xleftarrow{\;{}^tQ(x,D)\;} & \mathfrak{D}^{s_2} \end{array}$$

we have $R\circ{}^tP={}^tQ\circ S$, that is, $P(x,D)\,{}^tR(x,D)={}^tS(x,D)Q(x,D)$, hence a solution u of $Q(x,D)u=0$ corresponds to a solution $v={}^tR(x,D)u$ of $P(x,D)v=0$, and vice versa by choosing the commutative diagram to the opposite sense.

Usually we identify $\mathfrak{D}^{r_1}/{}^tP(x,D)\mathfrak{D}^{r_2}$ with the left $\mathfrak{D}$-module $\mathfrak{D}^{r_1}/\mathfrak{D}^{r_2}P(x,D)$ where $\mathfrak{D}^{r_2}$ is considered as the space of row vectors to which the matrix P operates from the right. Correspondingly we write, instead of (1.1),

$$(1.1)' \qquad 0\leftarrow\mathcal{M}\leftarrow\mathfrak{D}^{r_1}\xleftarrow{\;P(x,D)\;}\mathfrak{D}^{r_2}.$$

We should not forget that here $P(x,D)$ operates from the right.

The space of solutions of $\mathcal{M}$ of class $\mathcal{F}$ (where $\mathcal{F}=\mathcal{A}$ (the real analytic functions) or $\mathcal{B}$ (the hyperfunctions) or C^∞(infinitely differentiable functions) etc.) is intrinsically defined as $\mathcal{H}om_{\mathfrak{D}}(\mathcal{M},\mathcal{F})$. In fact, by the left exact property of the functor $\mathcal{H}om_D$ we have

$$(1.2) \qquad 0\to\mathcal{H}om_{\mathfrak{D}}(\mathcal{M},\mathcal{F})\to\mathcal{F}^{r_1}\xrightarrow{\;P(x,D)\;}\mathcal{F}^{r_2}.$$

Here $u={}^t(u_1,\cdots,u_{r_1})\in\mathcal{F}^{r_1}$, which corresponds to the element of

$\mathcal{H}om_{\mathfrak{D}}(\mathfrak{D}^{r_1},\mathcal{F})$ defined by $e_j \to u_j (1 \leq j \leq r_1)$ (e_j denoting the j-th unit vector of $\mathfrak{D}^{r_1}$), goes to $P(x,D)u$ calculated by the usual rule, which corresponds to the element of $\mathcal{H}om_{\mathfrak{D}}(\mathfrak{D}^{r_2}, \mathcal{F})$ defined by $e_j \to$ "the j-th component of $P(x,D)u$ $(1 \leq j \leq r_2)$". Thus we have $P(x,D)u=0$ just when the homomorphism $e_j \to u_j$ extends to an element of $\mathcal{H}om_{\mathfrak{D}}(\mathcal{M}, \mathcal{F})$, whence the exactness of the sequence (1.2).

Characteristic variety. To define the characteristic variety of a system $\mathcal{M}$ we consider the filtration $\{\mathfrak{D}_k\}$ of $\mathfrak{D}$ naturally defined by the order of differential operator:

$$\mathfrak{D}_k = \{p(x,D) \in \mathfrak{D};\ \text{order } (p(x,\ D)) \leq k\}.$$

The associated graded ring $\bigoplus_k (\mathfrak{D}_k/\mathfrak{D}_{k-1})$ is nothing but the ring of polynomials of ζ with coefficients in the sheaf $\mathcal{O}_{B_\delta}$ of commutative rings of germs of holomorphic functions on B_δ. To $\mathcal{M}$ we introduce the associated filtration $\{\mathcal{M}_k\}_{k=0}^{\infty}$ defined by

$$(1.3) \qquad \mathcal{M}_k = \mathfrak{D}_k^{r_1}/\mathfrak{D}^{r_1}P \cap \mathfrak{D}_k^{r_1}) \quad (k \geq 0).$$

We put $\mathcal{M}_k=0$ for $k<0$.

Definition 1.1. We call the analytic set defined as the support of the coherent $\mathcal{O}$-module on $B_\delta \times \boldsymbol{CP}^{n-1}$ associated to the graded $\mathcal{O}_{B_\delta}[\zeta]$-module $\mathrm{gr}\mathcal{M} = \bigoplus_k (\mathcal{M}_k/\mathcal{M}_{k-1})$ the *characteristic variety* or the *singular support* of $\mathcal{M}$ and denote it by S.S.$\mathcal{M}$.

In the sequel we consider S.S.$\mathcal{M}$ only as a set (i.e. without taking account of the multiplicity).

Lemma 1.2. *The analytic set*

$$(1.3) \qquad V_k = \bigcap_{f(x,\zeta)} \{f(x,\zeta)=0\},$$

where $f(x,\zeta)$ runs in the homogeneous polynomials satisfying $f\cdot(\mathcal{M}_k/\mathcal{M}_{k-1})=0$, is independent of k and agrees with S.S.$\mathcal{M}$.

In fact we have by definition

$$\text{S.S.}\mathcal{M} = \{(x,\zeta);\ f(x,\zeta)=0 \text{ for any homogenenous polynomial } f \text{ satisfying } f\cdot(\mathcal{M}_k/\mathcal{M}_{k-1})=0 \text{ for all } k\}.$$

Since $\mathcal{D}_1\mathcal{M}_k=\mathcal{M}_{k+1}$, $f\cdot(\mathcal{M}_k/\mathcal{M}_{k-1})=0$ implies $f\cdot(\mathcal{M}_{k+1}/\mathcal{M}_k)=0$ and conversely, $f\cdot(\mathcal{M}_{k+1}/\mathcal{M}_k)=0$ implies $\zeta_j f\cdot(\mathcal{M}_k/\mathcal{M}_{k-1})=0$, $j=1,\cdots,n$, whence $V_k=V_{k+1}=\mathrm{S.S.}\mathcal{M}$.

Remark 1. In order to facilitate the discussion, we usually consider a "good filtration" of $\mathcal{M}$ in general. It is a filtration $\{\mathcal{M}_k\}_{k=-\infty}^{\infty}$ of $\mathcal{M}$ satisfying the following conditions:

1) Each $\mathcal{M}_k$ is a coherent $\mathcal{O}_{B_\delta}$-module;

2) $\mathcal{M}=\bigcup_k \mathcal{M}_k$, $\mathcal{M}_k=0$ for $k\ll 0$;

3) $\mathrm{gr}\mathcal{M}=\bigoplus_k(\mathcal{M}_k/\mathcal{M}_{k-1})$ is a coherent $\mathcal{O}_{B_\delta}[\zeta]$-module.

Then we have (Kashiwara[1], Proposition 1.2.5).

4) $\mathcal{D}_1\mathcal{M}_k=\mathcal{M}_{k+1}$ for $k\gg 0$,

hence we can define S.S.$\mathcal{M}$ via any such filtration. However, we shall limit ourselves to the induced filtration because we do not intend the general theory.

S.S.$\mathcal{M}$ corresponds to the "part at infinity" of the notion of the characteristic variety for a system with constant coefficients (see e.g. Palamodov[1], Chapter IV, § 1). To show this analogy more clearly we choose a "good" representation of the system $\mathcal{M}$:

$$0\leftarrow\mathcal{M}\leftarrow\mathcal{D}^{s_1}\xrightarrow{\ Q\ }\mathcal{D}^{s_2} \tag{1.4}$$

such that, letting m to be the maximum value of orders of components of the matrix Q, the induced filtration $\mathcal{M}_k=\mathcal{D}_k^{s_1}/\mathcal{D}^{s_2}Q\cap\mathcal{D}_k^{s_1}$ satisfies

$$\mathcal{D}^{s_2}Q\cap\mathcal{D}_k^{s_1}=\mathcal{D}_{k-m}^{s_2}Q \quad \text{for } k\gg 0.$$

Then putting $Q_m(x,\zeta)$ to be the matrix of homogeneous polynomials constructed by the m-th order terms in Q, we have

$$\begin{aligned}(1.5)\qquad \mathrm{S.S.}\mathcal{M}&=\{(x,\zeta)\in B_\delta\times \boldsymbol{C}P^{n-1},\ \mathrm{rank}\ Q_m(x,\zeta)<s_1\}\\ &=\text{"the common zero points of } f(x,\zeta)\in\mathcal{O}_{B_\delta}[\zeta]\\ &\quad\ \text{satisfying } {}^tQ_m(x,\zeta)\mathcal{O}_{B_\delta}[\zeta]^{s_2}\supset f(x,\zeta)\mathcal{O}_{B_\delta}[\zeta]^{s_1}\text{"}.\end{aligned}$$

This process is a particular case of the so to speak "prolongation" of systems of differential equations to an involutory one: For simplicity assume that $\mathcal{M}$ is generated by only one element, or equivalently that there is given an initial representation of the form

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{D} \xleftarrow{P(x,D)} \mathcal{D}^r$$

with $P(x,D) = {}^t(p_1(x,D), \cdots, p_r(x,D))$, or that the given system of equations has the form

$$p_1(x,D)u = \cdots = p_r(x,D)u = 0. \tag{1.5}$$

Then the set V of common zeros of the principal symbols $p_j^0(x,\zeta)$ of $p_j(x,D)$, $j=1,\cdots,r$, does not necessarily give the correct definition of the characteristic variety of the system, because it does not necessarily satisfy the condition of involutivity:

$$f(x,\zeta) = g(x,\zeta) = 0 \text{ on } V \Rightarrow \{f(x,\zeta),\ g(x,\zeta)\} = 0 \text{ on } V, \tag{1.6}$$

where $\{,\}$ denotes the Poisson bracket (Recall that S.S.$\mathcal{M}$ is known to be always involutory: See S-K-K[1], Chapter II, Theorem 5. 3. 2.). If we add, however, some of equations deduced from (1.5), we always arrive to an involutory system which is equivalent to the original one and which in the same time supplies a "good" representation of the given system. For example, from the system $(D_1 + x_1 D_2)u = D_1^2 u = 0$ which is not involutory we arrive, after a little calculation, to an involutory system equivalent to $D_1 u = D_2 u = 0$.

Remark 2. It would be happy if S.S.$\mathcal{M}$ agreed with the analytic set defined by the ideal

$$\mathcal{J} = \{p_m(x,\zeta);\ \text{There exists } p(x,D) \text{ with the principal symbol } p_m(x,\zeta) \text{ such that } p(x,D)\mathcal{M} = 0\}.$$

However, this is true only when we employ the germs of micro-differential (or pseudo-differential) operators as $p(x,D)$. The sheaf $\mathcal{E}$ of micro-differential operators is a sheaf of rings on $B_\delta \times \boldsymbol{C}\boldsymbol{P}^{n-1}$ which "localizes" $\pi^{-1}\mathcal{D}$ ($\pi : B_\delta \times \boldsymbol{C}\boldsymbol{P}^{n-1} \to B_\delta$ is the canonical projection) : An operator $p(x,D) \in \mathcal{D}$ is invertible in $\mathcal{E}$ at (x,ζ) if its principal symbol $p_m(x,\zeta)$ does not vanish at this point. Thus S.S.$\mathcal{M}$ agrees with the sheaf theoretical support of the coherent $\mathcal{E}$-module $\mathcal{E} \otimes_{\pi^{-1}\mathcal{D}} \pi^{-1}\mathcal{M}$. In the sequel we also employ micro-differential operators as a convenient tool.

Later on we consider S.S.$\mathcal{M}$ also as a subset of $B_\delta \times (\boldsymbol{C} \times \boldsymbol{R}^{n-1})/\boldsymbol{R}^+$ without identifying the antipodal points. The sheaf $\mathcal{E}$ of micro-differential operators can also be considered on such a set.

Initial value to a non-characteristic surface.

Definition 1. 3. We say that a piece of hypersurface $S \subset \{x_1=0\}$ is *non-characteristic* with respect to a system $\mathcal{M}$ at a point $x^0 \in S$ if its conormal (x^0, ν), where $\nu=(1,0,\cdots,0)$, does not belong to S.S.$\mathcal{M}$. We say that S is non-characteristic if it is non-characteristic at every point. We have

Lemma 1. 4. *Assume that S is non-characteristic with respect to $\mathcal{M}$. Then there exists a homogeneous polynomial $f(x,\zeta) \in A(B_\delta)[\zeta]$ of ζ such that $f(x,\nu) \neq 0$ and that $f \cdot (\mathcal{M}_k/\mathcal{M}_{k-1})=0$ for any k.*

In fact such an f exists by the definition locally at every $x^0 \in B_\delta$. Since B_δ is Stein f may be assumed to be defined on B_δ. We pick up a finite number of such f's so that at any $x^0 \in B_\delta$ one of them does not vanish at (x^0, ν). Then we adjust their order to a common m, and finally take the sum of their squares.

Proposition 1. 5. *Assume that S is non-characteristic with respect to $\mathcal{M}$. Then there exists a homogeneous polynomial $f(x,\zeta) \in \mathcal{A}(B_\delta)[\zeta]$ such that $f(x,\nu)=1$ and that*

$$f(x,D)e_i = \sum_{j=1}^{r_1} b_{ji}(x,D)e_j \quad \text{in } \mathcal{M}, \tag{1.7}$$

where e_j $(1 \leq j \leq r_1)$ denotes the generators of $\mathcal{M}$ in the represetantion (1. 1) *and* order $(b_{ji}) < m = \text{order}(f)$.

In fact, it suffices to choose $f(x,\zeta)$ as in the above Lemma (after dividing it by the coefficient of D_1^m) and interpret the identity $f \cdot (\mathcal{M}_1/\mathcal{M}_0)=0$.

Corollary 1. 6. *If S is non-characteristic with respect to $\mathcal{M}$, then $\mathcal{M}|_s$ becomes a finitely generated $\mathcal{O}_{B_\delta \cap S}[D_2, \cdots, D_n]$-module.*

In fact, the stalk of $\mathcal{M}$ at every point of S is generated over

$\mathcal{O}_{B_\delta}[D_2, \cdots, D_n]$ by $D_1^k e_j (1 \leq j \leq r_1,\ 0 \leq j \leq m-1)$ as is seen form (1.7).

Definition 1.7. Assume that S is non-characteristic with respect to $\mathcal{M}$. Then we put $\mathcal{N} = {}'\mathcal{O} \otimes_{\mathcal{O}} \mathcal{M}$ (where $'\mathcal{O}$ is considered as a module over $\mathcal{O} = \mathcal{O}_{B_\delta}$ by the action $x_1 \cdot {}'\mathcal{O} = 0$), consider it a module over $'\mathfrak{D} = {}'\mathcal{O}[D_2, \cdots, D_n]$ and call it the *induced system* of $\mathcal{M}$ to S.

By the above discussion $\mathcal{M}$ becomes a system of differential equations in the variables $x' = (x_2, \cdots, x_n)$ on S.

Defnition 1.8. The natural mapping

$$(1.8) \qquad \gamma : \mathcal{H}om_{\mathfrak{D}}(\mathcal{M}, \mathcal{O})\,|_S \longrightarrow \mathcal{H}om_{'\mathfrak{D}}(\mathcal{N}, {}'\mathcal{O})$$

is called the initial value operation. The image of $u \in \mathcal{H}om_{\mathfrak{D}}(\mathcal{M}, \mathcal{O})$ by this mapping is called the *initial value* of the holomorphic solution u.

From Corollary 1.6 we have the following represention for the induced system $\mathcal{N}$:

$$(1.9) \qquad \begin{array}{ccccccc} 0 \leftarrow & \mathcal{M} & \longleftarrow & \mathfrak{D}^{r_1} & \xleftarrow{\qquad P(x,D) \qquad} & \mathfrak{D}^{r_0} \\ & \wr\uparrow & & \uparrow & & \Big| \\ 0 \leftarrow & \mathcal{N} & \longleftarrow & \prod_{j=1}^{r_1} \bigoplus_{k=0}^{m-1} {}'\mathfrak{D}^{r_1} D_1^k & \xleftarrow{\qquad Q(x',D') \qquad} & {}'\mathfrak{D}^l \end{array}$$

(Here the first row is also considered as an exact sequence of $\mathcal{O}[D_2, \cdots, D_n]$-modules. In the second row $x_1 \cdot$ acts trivially. The choice of Q or of the last vertical arrow to make the diagram commutative is not unique.). Then the initial value of $u \in \mathcal{H}om_{\mathfrak{D}}(\mathcal{M}, \mathcal{O})$ is nothing but $\gamma u = (D_1^k u|_{x_1=0})_{0 \leq k \leq m-1}$. It satisfies the induced system of differential equations $Q(x',D')\gamma u = 0$ which arises naturally as the relation among the components of the initial value from the original system of equations $Pu = 0$. This notion evidently generalizes the usual initial value to a non-characteristic hypersurface for a single equation of order m. In that case the induced system is isomorphic to $'\mathfrak{D}^m$ and there arise no relations among the components of γu. More generally we have obviously.

Lemma 1.9. *The induced system of the system defined by the relation* (1.7) *to* S *is isomorphic to the free module* $'\mathfrak{D}^{mr_1}$.

Of course a solution u of the original system $\mathcal{M}$ satisfies equations other than those corresponding to (1.7), thus the initial value γu obeys some relations in general.

Theorem 1.10 (Kashiwara's Cauchy-Kowalewsky theorem; Kashiwara[1], Theorem 2.3.1). *The mapping* γ *in* (1.8) *is an isomorphism.*

The initial value to a non-characteristic hypersurface S has a meaning also for the hyperfunction solutions:

$$(1.8)' \qquad \gamma: \mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{B})|_S \longrightarrow \mathcal{H}om_{\mathcal{D}'}(\mathcal{M},'\mathcal{B}).$$

We can legitimately consider the data $D_1^k u|_{x_1=0}$ because in view of (1.7) $u=(u_i)$ satisfies the system of differential equations

$$(1.10) \qquad f(x,D)u_i=\sum_{j=1}^{r_1} b_{ji}(x,D)u_j, \quad 1\leq i\leq r_1,$$

hence each component u_i satisfies a micro-differential equation with the principal symbol $f(x,\zeta)$ which is micro-elliptic on a neighborhood of $(B_\delta\cap S)\times\{\nu\}$. This time γ is always injective in view of Holmgren's theorem but becomes isomorphic only when $\mathcal{M}$ is hyperbolic, i.e., when $(x;\zeta)\in$S.S.$\mathcal{M}$, $\zeta'\in \boldsymbol{R}^{n-1}$ implies $\zeta_1\in\boldsymbol{R}$ (see Kashiwara-Schapira[1], Corollary 2.3.2.).

2. Singular spectrum of boundary values of solutions of a system.

Boundary value to a non-characteristic hypersurface. The data $D_1^k u|_{x_1=0}$ are even legitimate if only u is a "mild" hyperfunction on $x_1>0$, i.e., if u admits a boundary value representation

$$(2.1) \qquad u(x)=\sum_{j=1}^{N} F_j(x+i\Gamma_j)$$

such that each $F_j(z)$ can be extended holomorphically to a domain of the form

$$(2.2) \qquad \{z=x+iy;\ (-x_1)_+ +\lambda_j|y_1|<|y'|<\delta_j,\ y'\in\Delta_j,\ x\in B_\delta\}$$

where $\Delta_j\subset\boldsymbol{R}^{n-1}$ is a proper cone such that $\Gamma_j\supset\{y\in\boldsymbol{R}^n;\ \lambda_j|y_1|<|y'|,\ y'\in\Delta_j\}$. Then

(2. 3) $$D_1^k u|_{x_1\to+0}=\sum_{j=1}^{N}(D_1^k F_j)(0, x'+i\Delta_j 0)$$

(See Kataoka[2], Definition 2. 1. 1 and Corollary 2. 1. 5 for the precise description of these notions.). Recall that in the case of initial value the micro-analyticity of u to the direction ν implies that u has a representation (2. 1) with F_j holomorphic in

(2. 2)′ $$\{z=x+iy;\ \lambda_j|y_1|<|y'|<\delta_j,\ y'\in\Delta_j,\ x\in B_\delta\}$$

and $D_1^k u|_{x_1=0}$ is given by the same formula as (2. 3).

Proposition 2. 1. *Let u be a solution of $\mathcal{M}$ defined on $B_\delta\cap\{x_1>0\}$. Assume that S is non-characteristic with respect to $\mathcal{M}$. Then u is mild, hence we can consider the boundary value $D_1^k u|_{x_1\to+0}$.*

Proof. Since u satisfies the system (1. 10), it suffices to prove the assertion for this special system. Choose an extension $[u_j]$ of u_j with support in $x_1\geq 0$. Then we have

(2. 4) $$f(x,D)[u_i]=\sum_{j=1}^{r_1} b_{ji}(x,D)[u_j]+v_i,\quad 1\leq i\leq r_1,$$

where v_i is a hyperfunction with support in $x_1=0$. By Lemma 2. 2 below we can modify $[u_j]$ so that v_i has the form

$$\sum_{k=0}^{m-1} v_{ik}(x')\delta^{(k)}(x_1).$$

Since $[u_j]$ can be obtained from these mild hyperfunctions by the inversion of a micro-differential operator which is micro-elliptic on a neighborhood of the direction ν, they become also mild as is easily seen by the elementary description of the action of the pseudo-differential operator given in Bony-Schapira[1]. (More advanced proof will be found in Kataoka[2], Example 2. 1. 3.). Q.E.D.

Recall that $[u_i]$ above is the so called canonical (flabby) extension of u_i and agrees with the product $Y(x_1)u_i$ which is naturally defined in view of the form of the domain (2. 2) (Kataoka [2], Remark 2. 2. 4).

The following lemma can be obtained by the same argument as in Komatsu-Kawai[1] for the case of a single equation:

Lemma 2.2. *Let* v_i, $i=1,\cdots,r_1$, *be hyperfunctions with support in* $x_1=0$. *Then we can find hyperfunctions* w_i, $i=1,\cdots,r_1$ *with support in* $x_1=0$, *hyperfunctions* v_{ik} *of variables* x' *such that*

$$(2.5)\qquad f(x,D)w_i=\sum_{j=1}^{r_1} b_{ji}(x,D)w_j+\sum_{k=0}^{m-1} v_{ik}(x')\delta^{(k)}(x_1)-v_i.$$

These hyperfunctions are determined from v_i *locally and uniquely along* $x_1=0$.

In fact, it suffices to solve the same "division problem" for v_i with compact support without enalarging the support. This is nothing but the dual assertion of the classical Cauchy-Kowalewsky theorem for the dual system of (1.7) (hence easier than the general theorem 1.10).

Corollary 2.3. *Assume that S is non-characteristic with respect to the system* $\mathcal{M}$. *Then* $\mathcal{M}$ *has no non-trivial solution whose support is contained in S.*

In fact such a solution u also satisfies (1.7), hence reduces to zero by the uniqueness of the expression (2.5).

Proposition 2.4. *We have the canonical mapping of boundary value*

$$(2.6)\qquad \gamma_+:\Gamma(B_\delta\cap\{x_1>0\},\ \mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{B}))\to\Gamma(B_\delta\cap\{x_1=0\},\ \mathcal{H}om_{'\mathcal{D}}(\mathcal{N},'\mathcal{B})).$$

Proof. Put $\gamma_+u=(D_1^ku|_{x_1\to+0})_{0\le k\le m-1}$, where $D_1^ku|_{x_1\to+0}$ is given by (2.3). However we cannot choose F_j in (2.1) among the holomorphic solutions of the given system: The fact that $Pu=0$ as hyperfunctions only implies that

$$(2.7)\qquad PF_j=\sum_{k=1}^{N} G_{jk},$$

where $G_{jk}(z)$ are anti-symmetric in j,k and each holomorphic on a domain (2.2) where Δ_j is replaced by $\Delta_j+\Delta_k$ (and the other constants a little diminished). (Edge of the wedge theorem for mild hyperfunctions; see Kataoka[2], Theorem 2.1.23.) Consider the following diagram which is obtained form (1.9) by taking the dual:

$$\begin{array}{ccc} \mathcal{O}^{r_1} & \xrightarrow{\quad P(x,D) \quad} & \mathcal{O}^{r_2} \\ \downarrow & & \Big| \\ \prod_{j=1}^{r_1} \bigoplus_{k=0}^{m-1} {}'\mathcal{O}^{r_1} & \xrightarrow{\quad Q(x',D') \quad} & {}'\mathcal{O}^{l}. \end{array}$$

Thus we see that $Q(x',D')\gamma F_j$ is not necessarily individually equal to zero but has the form of the right hand side of (2.7) in total (by the terms resulting from G_{jk}'s via the right vertical arrow). Thus the hyperfunctions $Q(x',D')\gamma_+u$ determined by them is zero, i.e., γ_+u satisfies the induced system of differential equations. We omit to show the canonicality of the mapping γ_+ since we do not require it (see e.g. S-K-K[1], Chapter II, Corollary 3.5.8). Q.E.D.

Singular spectrum of boundary values. To estimate the singular spectrum (S.S. for short in the sequel) of the boundary value of solutions of a single equation, we have introduced in Kaneko[3] the notion of boundary characteristic points which may be translated as follows in the case of a system:

Definition 2.5. Let $S=\{x_1=0\}$ be non-characteristic with respect to $\mathcal{M}$. Then we put

$$V^+_{S,\mathcal{A}}(\mathcal{M})=\{(x',\zeta')\in S\times S^{n-2}; \text{There exists a sequence } (x^{(k)},\zeta^{(k)})$$

$$\in \text{S.S.}\mathcal{M} \text{ such that } x_1^{(k)}>0,\ \zeta'^{(k)}\in R^{n-1},$$

$$\text{Im}\zeta_1^{(k)}>0 \text{ and } x^{(k)}\to(0,x'),\ \zeta'^{(k)}\to\zeta'\}.$$

We define $V^+_{S,\mathcal{B}}(\mathcal{M})$ by replacing the condition $\text{Im}\zeta_1^{(k)}>0$ by $\text{Im}\zeta_1^{(k)}\geq 0$ in the above. We call them the set of $\mathcal{A}$—(resp. $\mathcal{B}$—) *boundary characteristic points* of the system $\mathcal{M}$ from the *positive* side of the hypersurface S.

We expect that we may obtain the same results as in the case of a single equation, i.e., $\text{S.S.}\gamma_+u\subset V^+_{S,\mathcal{A}}(\mathcal{M})$ (resp. $V^+_{S,\mathcal{B}}(\mathcal{M})$) for a real analytic (resp. hyperfunction) solution u on $x_1>0$. Since the analysis of characteristic variety of a system is rather complicated, we content ourselves here by partial results as follows.

Let $\rho : S^{n-1}/\{\pm\nu\}\rightarrow S^{n-2}$ denote the canonical projection to the equator. We also let ρ denote similar projections such as $CP^{n-1}/\{\pm\nu\}\rightarrow CP^{n-1}$ or their extension to the product space $B_\delta\times(S^{n-1}/\{\pm\nu\})\rightarrow B_\delta\times S^{n-1}$ etc. We have first of all.

Proposition 2.6. *For any hyperfunction solution u of $\mathcal{M}$ on $x_1>0$,* S.S.γ_+u *is contained in* S.S.$\mathcal{N}$.

In fact, if $(x'^0,\xi'^0)\notin$S.S.$\mathcal{N}$. then γ_+u becomes microanalytic there as a solution of the system $\mathcal{N}$ (Corollary 3.2 and Sato's fundamental theorem).

Recall that (cf. Kashiwara-Schapira[1], § 1.4)

$$\text{S.S.}\mathcal{N}=\rho(\text{S.S.}\mathcal{M}|_S),$$

where now $\rho : (B_\delta|_S)\times(CP^{n-1}/\{\pm\nu\})\longrightarrow(B_\delta\cap S)\times CP^{n-1}$. Thus the above proposition concerns the points (x'^0,ξ'^0) for which $\rho^{-1}(x'^0,\xi'^0)\cap$S.S.$\mathcal{M}=\phi$. We must study the case where this set is not void but satisfies the condition of "semi-hyperbolicity".

Theorem 2.7. *Let (x'^0,ξ'^0) be a point of the cosphere bundle of the non-characteristic boundary $S=\{x_1=0\}$. Assume that*

i) *for (x,ξ') in the set*

(2.8) $\{(x,\xi');x\in\mathbf{R}^n,\ x_1\geq 0,\ |x-x'^0|<\varepsilon,\ \xi'\in\mathbf{R}^{n-1},\ |\xi'-\xi'^0|<\varepsilon\}$

$\rho^{-1}(x,\xi')\cap$S.S.$\mathcal{M}$ *always consists of points (x,ζ_1,ξ') satisfying* $\mathrm{Im}\zeta_1\leq 0$ *(where now $\rho : B_\delta\times((\mathbf{C}\times\mathbf{R}^{n-1})/\mathbf{R}^+/\{\pm\nu\})\rightarrow B_\delta\times S^{n-2}$);*

ii) *Every germ of irreducible component V of* S.S.$\mathcal{M}$ *touching the meridian $\rho^{-1}(x'^0,\xi'^0)$ is proper and finite over its projection image $V'=\rho(V)$, and V' is non-singular at (x'^0,ξ'^0) and projectable to the x-space* (see (2.10) below for the concrete meaning of these conditions).

Then for every real analytic solution u of $\mathcal{M}$ on $x_1>0$, γ_+u becomes micro-analytic at (x'^0,ξ'^0).

Note that the set of points (x'^0,ξ'^0) which do not satisfy the condition i) is nothing but $V^+_{S,\mathcal{A}}(\mathcal{M})$. We need a technical condition ii) in addition whose role will be clarified in the proof below (See also Remark after the proof.).

Proof. This result has been proved in Kaneko[3], Theorem 2.1 for a single differential equation. It has been generalized in Kataoka[3], Theorem 1.12 to a micro-local solution at the boundary for a micro-differential equation. Note that for a micro-differential operator $p(x,D)$, which is a differential operator in D_1, we consider from the beginning a mild hyperfunction u which satisfies

$$p(x,D)(Y(x_1)u)=\sum_{k=0}^{m-1} v_k(x')\delta^{(k)}(x_1)+Y(x_1)w,$$

with a mild hyperfunction w which is micro-analytic on a neighborhood of the meridian $\rho^{-1}(x'^0,\xi'^0)$ under consideration.

Returning to our case we prove below that we can choose a micro-differential operator $f(x,D)$ which satisfies a relation such as (1.7) in the sense of micro-differential operators on a neighborhood of $\rho^{-1}(x'^0,\xi'^0)$ so that the roots of $f(x,\zeta)=0$ for ζ_1 satisfy $\mathrm{Im}\,\zeta_1\leq 0$ for $x_1\geq 0$ on a neighborhood of (x'^0, ξ'^0). Then we can retrace the proof in the case of a single micro-differential equation, because the dual system of (1.10) has the same form and we can solve the "unilateral Cauchy problem" for the latter with the special Cauchy data made of the component of a curved Radon decomposition of $\delta(x')$ and then apply the Green formula just as in the case of a single micro-differential equation.

Now we try to find such a polynomial $f(x,\zeta)$ of ζ_1 as above. We can put $(x'^0, \xi'^0)=(0, v')$, where $v'=(1, 0, \cdots, 0)$. Thus we shall find a homogeneous function $f(x,\zeta)$ of ζ which is a monic polynomial in ζ_1 with the coefficients in $\boldsymbol{C}\{x, \zeta_3/\zeta_2, \cdots, \zeta_n/\zeta_2\}[\zeta_2]$ (where $\boldsymbol{C}\{\cdots\}$ denotes the ring of the convergent power series of the indicated variables at 0) such that $f\cdot(\mathcal{M}_k/\mathcal{M}_{k-1})=0$ for all k in the scalar extension of the corresponding module (which we are denoting by the same letter for simplicity) and that the variety $f(x,\zeta)=0$ does not intersect the set

$$\{(x,\zeta_1,\xi')\ ;\ x_1\geq 0,\ |x'|<\varepsilon,\ \mathrm{Im}\,\zeta_1>0,\ \xi'\in\boldsymbol{R}^{n-1},\ |\xi'-v'|<\varepsilon\} \tag{2.9}$$

for some smaller $\varepsilon>0$. In view of the Hilbert Nullstellensatz we can replace the condition $f\cdot(\mathcal{M}_k/\mathcal{M}_{k-1})=0$ by the vanishing of f on S.S.$\mathcal{M}$ on a neighborhood of $\rho^{-1}(0, v')$. Further, we can find such a polynomial to every germ of irreducible components of S.S.$\mathcal{M}$ over the point $(0, v')$ and employ the product of them. By the assumption ii) every irreducible component of S.S.$\mathcal{M}$ is expressed in the form

$$(2.10)\qquad \{(x,\zeta)\,;\, f(x,\zeta)=0,\ (x,\zeta')\in V'\},$$
$$V'=\{(x,\zeta')\,;\, f_1(x,\zeta')=0,\cdots,f_N(x,\zeta')=0\}\ (N\leq n-2),$$

where f is a monic polynomial in ζ_1 with coefficients in the ring $C\{x,\zeta_3/\zeta_2,\cdots,\zeta_n/\zeta_2\}[\zeta_2]$, homogeneous in ζ in total (assumption of properness and finiteness), and f_j are elements of this ring whose differentials with respect to ζ' are linearly independent at $(0,v')$ (assumption of non-singularity and projectability of V'). Making a linear coordinate transformation and dividing by a non-vanishing factor if necessary we can assume that each f_j is homogeneous of degree 1 in ζ' and of the form

$$f_j(x,\zeta')=\zeta_{j+2}-\zeta_2 g_j(x,\zeta_{j+3}/\zeta_2,\cdots,\zeta_n/\zeta_2)$$

where g_j are germs of holomorphic functions at $(0,0)$, homogeneous of degree 0 in the latter variables. By the main assumption i) of the theorem the roots of the equation $f(x,\zeta)=0$ for ζ_1 all satisfy $\operatorname{Im}\zeta_1\leq 0$ for real (x,ζ') in the region (2.8) provided $f_j(x,\zeta')=0$, $j=1,\cdots,N$. We must modify $f(x,\zeta)$ to $g(x,\zeta)$ without changing the roots for $(x,\zeta')\in V'$ so that the roots always satisfy $\operatorname{Im}\zeta_1\leq 0$ for real (x,ζ') in (2.8). However, under our assumption such a modification can be easily obtained by substituting $\zeta_j=\zeta_2 g_j(x,\zeta_{j+1}/\zeta_2,\cdots,\zeta_n/\zeta_2)$, $j=3,\cdots,N+2$ to the coefficients of $f(x,\zeta)$. That this $f(x,D)$ satisfies a relation like (1.7) follows from an argument analogous to Proposition 1.5. Q.E.D.

Remark. It would be happy if we could find a homogeneous single differential operator $f(x,D)$ as in (1.7) such that its characteristic variety satisfies the assumption of the theorem in itself without the technical assumption ii). However, this is not true in general even for a system with constant coefficients even if we allow f to be a micro-differential operator. Consider e.g. the following system on $\boldsymbol{R}^5$ on a neighborhood of $\xi'=v'=(1,0,0,0)$:

$$(2.11)\qquad \begin{cases} (D_1^2-D_2D_3)u=0, \\ (D_2D_3^3-D_4^4-D_5^4)u=0. \end{cases}$$

Put $V'=\{\xi_2\xi_3^3=\xi_4^4+\xi_5^4\}$. The characteristic variety or rather the set of characteristic roots

$$\{(\zeta_1,\xi');\xi'\in V',\ \zeta_1=\pm\sqrt{\xi_2\xi_3}\}$$

is real for real $\xi'\sim\nu'$ because $\xi_2>0$ hence $\xi_3\geq 0$ by the condition $\xi'\in V'$. Any monic polynomial $f(\zeta)$ of ζ_1 whose roots agree with $\pm\sqrt{\xi_2\xi_3}$ on V' must contain a homogeneous factor of the second degree

$$(2.12)\qquad \zeta_1^2-2a(\xi')\zeta_1-b(\xi')$$

such that $b(\xi')|_{V'}=\xi_2\xi_3$ and $a(\xi')|_{V'}=0$, because $\sqrt{\xi_2\xi_3}|_{V'}$ is not an element of the ring $C[\xi']/(\xi_2\xi_3^3-\xi_4^4-\xi_5^4)$. However there are no such polyonmials other than $a(\xi')=0$ and $b(\xi')=\xi_2\xi_3$. Introduction of the use of a micro-differential operator f only allows us to employ a function of $(\xi_2\xi_3^3-\xi_4^4-\xi_5^4)/\xi_2^3$. Since this is of order $O(\xi_3^3)$ for $\xi_4=\xi_5=0$, it is impossible to find a modification such that the polynomial (2.12) always has real roots as ξ_3 changes the sign (Note that the operator $D_1^2-\sqrt[3]{D_2^2(D_4^4+D_5^4)}$, as naturally occurs by the elimination, is not in the well studied classes of analytic pseudo-differential operators.). For this system it is not at all obvious if the conclusion of Theorem 2.7 holds as well. Remark that this system is not micro-hyperbolic at $(0,\nu')$ to the direction ν in the sense of Kashiwara-Schapira[1].

The following theorem can be proved just similarly as above. (Kataoka[3] does not consider the generalization of Theorem 2.2 in Kaneko[3]. But it can be made parallelly.)

Theorem 2.8. *Let* (x'^0,ξ'^0) *be a point of the cosphere bundle of the non-characteristic boundary* S. *Assume that*

i) *for* (x,ξ') *in the set*

$$(2.13)\qquad \{(x,\xi');\ x\in R^n,\ x_1>0,\ |x-x'^0|<\varepsilon,\ \xi'\in R^{n-1},|\xi'-\xi'^0|<\varepsilon\}$$

the fiber $\rho^{-1}(x,\xi')\cap \mathrm{S.S.}\mathcal{M}$ *consists of points* (x,ζ_1,ξ') *satisfying*

$\mathrm{Im}\zeta_1<0$;

ii) *the same condition as in Theorem* 2.7 *holds.*

Then for every hyperfunction solution u *of* $\mathcal{M}$ *on* $x_1>0$, γ_+u *becomes micro-analytic at* (x'^0, ξ'^0).

Note again that the set of points (x'^0, ξ'^0) which do not satisfy the condition i) is nothing but $V^+_{S,\mathcal{B}}(\mathcal{M})$. The same remark applies as for the meaning of the technical condition ii).

To discuss the continuation of solutions in the next section, we also need to consider the boundary value of the solution from the negative side of S. The set of $\mathcal{A}$- (resp. $\mathcal{B}$-) *boundary characteristic points* $V^-_{S,\mathcal{A}}(\mathcal{M})$ (resp. $V^-_{S,\mathcal{B}}(\mathcal{M})$) of a system $\mathcal{M}$ from the *negative* side of S is defined by reversing all the inequality signs appearing in Definition 2.5. Equally Theorems 2.7, 2.8 may be translated by the same procedure, and we thus obtain results giving the estimation of S.S. of the boundary value γ_-u of a real analytic (resp. hyperfunction) solution u on $x_1<0$, which we shall refer as Theorem 2.7′ (resp. Theorem 2.8′) in the sequel.

3. Extension of solutions of a system.

The argument employed in Kaneko[3] gives equally the following.

Theorem 3.1. *Let* K *be a subset of* $S=\{x_1=0\}$ *which is contained in one side of a hypersurface* $\phi(x')=0$ *of* S *of class* C^1 *such that either of the points* $(0, \pm id\phi(x')\infty)$ *satisfies the condition of Theorem* 2.7 *and of Theorem* 2.7′ (*resp.* 2.8 *and* 2.8′). *Then every real analytic* (*resp. hyperfunction*) *solution* u *of* $\mathcal{M}$ *defined in* B_δ/K *can be uniquely continued as a hyperfunction solution to a neighborhood of the origin.*

Proof. u can be considered as a solution of $\mathcal{M}$ on $B_\delta\cap\{\pm x_1>0\}$, hence mild from both sides of S. By the Leibniz rule we have

$$P(x,D)Y(\pm x_1)u=\pm\sum_{k=0}^{m-1}u_k(x')\delta^{(k)}(x_1),$$

where m is the maximal value of the components of P and each

$u_k^{\pm}(x')$ is a vector of hyperfunctions on S which can be expressed by the components of the boundary value $\gamma_{\pm}u$ of u via a differential operator on S. We first show that $\mathrm{supp}(u_k^+(x')-u_k^-(x'))$ is contained in K. In fact, outside K we have

$$0=P(x,D)u=P(x,D)\,(Y(x_1)u+Y(-x_1)u)$$
$$=\sum_{k=0}^{m-1}(u_k^+(x')-u_k^-(x'))\delta^{(k)}(x_1),$$

hence we have $u_k^+-u_k^-=0, k=0,\cdots,m-1$, successively. Next we remark that $\mathrm{S.S.}(u_k^+-u_k^-)$ is free from either of the points $((0,\pm id\phi(x')\infty)$ by Theorems 2.7 and 2.7′, and the corresponding assumption. In view of the Kashiwara-Kawai Holmgren type theorem these two informations imply that $u_k^+-u_k^-=0$ on a neighborhood of the origin. Thus $Y(x_1)u+Y(-x_1)u$ satisfies the equation $Pu=0$ on this neighborhood and serves as an extension of the original solution to there.

Finally the uniqueness of the extension follows from Corollary 2.3. Q.E.D.

Corollary 3.2. *Assume that there exists linearly independent vectors* $v,\xi^0\in\boldsymbol{R}^n$ *such that* $\mathcal{M}$ *is hyperbolic to the direction* v *at* $(0,\xi^0)$ *in the sense that*

$$(x,v)\notin \mathrm{S.S.}\mathcal{M},\quad (x,\tau v+\xi)\notin \mathrm{S.S.}\mathcal{M}$$

for $x\sim 0$, $\xi\sim\xi^0$ *and* $\mathrm{Im}\,\tau\neq 0$, *and that every irreducible germ of* S.S. $\mathcal{M}$ *touching the meridian* $\{(0,tv+\xi);t\in\boldsymbol{R}\}$ *is nonsingular, projectable to the x-space and is placed transversally to the direction* v. *Then every real analytic solution of* $\mathcal{M}$ *defined outside the origin can be uniquely continued to the origin as a hyperfunction solution.*

Here a variety $V\subset B_\delta\times\boldsymbol{CP}^{n-1}$ is called projectable to the x-space if V is transversal to $\pi: B_\delta\times\boldsymbol{CP}^{n-1}\to B_\delta$, i.e., if π is surjective and $d\pi$ has maximal rank on V (Recall that the dimension of every irreducible component of S.S.$\mathcal{M}$ is not less than n because of the involutive property.).

This corollary reduces to a particular case of Theorem 3.1 if we choose the system of coordinates such that $\upsilon=\nu$ (The condition of Theorem 3.1 itself is rather difficult to express intrinsically.). Thus we have at least succeeded to generalize the result of Kaneko[2] (the part concerning real analytic solutions) to the case of systems. Note that in the case of variable coefficients we cannot assert to obtain the extension in real analytic solutions in general even for a determined system. This is a problem on the propagation of regularity. See kawai[1], Theorem 3 for a counterexample. Sufficient condition for the non-existence of a solution with an isolated singular support seems unknown even for a single equation except for some obvious ones such as micro-hyperbolicity at every point $(0,\xi)$ (see Kashiwara-Schapira [1], Theorem 2.2.3).

Corollary 3.3. *Assume that the induced system of* $\mathcal{M}$ *to* S *has at least one non-characteristic direction* ξ' *at the origin. Then every hyperfunction solution of* $\mathcal{M}$ *defined outside the origin can be uniquely continued to the origin.*

This result follows directly from Proposition 2.1 though it is anyway a particular case of Theorem 3.1.

Corollary 3.3 may be considered as the correspondent of the unique continuation theorem of hyperfunction solutions to an isolated point for a determined and overdetermined system with constant coefficients. In fact we have

Proposition 3.4. *Assume that* $\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{D})=\mathcal{E}xt^1_{\mathcal{D}}(\mathcal{M},\mathcal{D})=0$ *and that each component of* S.S.$\mathcal{M}$ *is projectable to the* x*-space. Then* $\mathcal{M}$ *satisfies the condition of Corollary* 3.3 *with a suitable choice of the system of coordinates.*

In fact, just as in the case of constant coefficients we have

$$\text{S.S.}\mathcal{M}=\bigcup_{i=0}^{n}\text{S.S.}\,(\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{N}.\mathcal{D}.)),\qquad \text{codim S.S.}\,(\mathcal{E}xt^i_i(\mathcal{M},\mathcal{D}))\geq i$$

(see Kashiwara[1], Proposition 3.2.7 and Theorem 3.1.3). Thus under our assumption the fiber S.S.$\mathcal{M}\cap\{0\}\times \boldsymbol{C}P^{n-1}$ of S.S.$\mathcal{M}$ at 0 is a projective variety of codimension≥ 2, hence we can choose a suitable real coordinate transformation such that $(0,\nu)\notin$S.S.$\mathcal{M}$. Then the induced system $\mathcal{N}$ has the characteristic variety $S.S.\mathcal{N}=\rho(S.S.\mathcal{M})$ whose fiber at 0 is a projective variety of codimension ≥ 1. Thus $\mathcal{N}$ has a non-characteristic direction ξ' at 0.

Remark. Contrary to the case of constant coefficients the $\mathfrak{D}$-module $\mathcal{B}[\{0\}]$ of hyperfunctions with the isolated support 0 is not flat, and the $\mathfrak{D}$-module $\mathcal{B}_0$ of the germs of hyperfunctions at 0 is not injective. Thus we cannot follow the general cohomological argument on the removability of the isolated singularity of hyperfunction solutions for a system with constant coefficients. For example, for the $\mathfrak{D}$-module $\mathcal{M}=\mathfrak{D}/\mathfrak{D}x_1$, which corresponds to the trivial "system" of equations $x_1 \cdot u=0$, we have

$$\mathcal{H}om_{\mathfrak{D}}(\mathcal{M},\mathfrak{D})=0 \quad \text{and} \quad \mathcal{E}xt^1_{\mathfrak{D}}(\mathcal{M},\mathfrak{D})=\mathfrak{D}/x_1\mathfrak{D}\neq 0.$$

But we can obviously extend any hyperfunction solution of $\mathcal{M}$ defined outside the origin up to the origin and the extension is not unique (Of course $\mathcal{H}om_{\mathfrak{D}}(\mathcal{M},\mathcal{B})[\{0\}])=0$ or $\mathcal{E}xt^1_{\mathfrak{D}}(\mathcal{M},\mathcal{B}[\{0\}])=0$ is a tortological condition and has no utility.). Cf, Kawai[1], Theorem 1 where he gives a sufficient condition for the removability of an isolated singularity of a hyperfunction solution on posing conditions to all the modules $\mathcal{E}xt^i_{\mathfrak{D}}(\mathcal{M},\mathfrak{D})$ which facilitate to construct the dual system. This type of assumption will be also useful to us to proceed by generalizing Green's formula to a general overdetermined system and giving a direct proof instead of reducing to a single equation.

The study of underdetermined case will be discussed elsewhere.

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